SEPARABLE DERANGEMENTS

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Dartmouth College

joint work with Robert Dougherty-Bliss, Alejandro Galvan, and David Shuster

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22 July 2026

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Separable Permutations

Av(2413,3142)

Derangements

no fixed points

Separable Permutations

Av(2413,3142)

A006318
Large Schröder numbers

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A000166
Subfactorial numbers

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D-finite generating function

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Enumerated using geometric decomposition

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Enumerated using geometric decomposition	?? No obvious approach to enumeration	Enumerated using inclusion-exclusion

Theorem (Brignall, Huczynska, Vatter 2007)

Given a class of permutations $\mathcal{C}$ with only finitely many simple permutations and some "nice" property $\mathcal{P}$, the generating function for the set of permutations in $\mathcal{C}$ satisfying $\mathcal{P}$ is algebraic.

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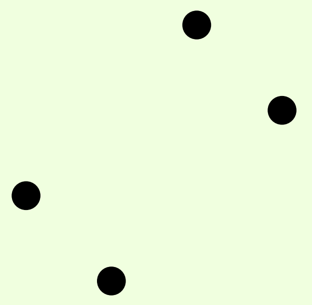
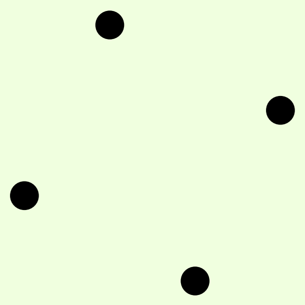
Definition

A permutation π is *simple* if it has no non-trivial intervals; i.e. $[a, b] \subset [1, n]$ such that $\{\pi(i) : a \leq i \leq b\}$ is contiguous.



2143
not simple



2413
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✓ Separable permutations: only finitely many simple perms $(1,\ 12,\ 21)$

✗ Derangements: not a "nice" property

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The first few terms are 0, 1, 2, 7, 30, 124, 560, 2610, 12470, ... These are not in the OEIS.

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Interesting, let me ask Jay to compute some more terms and see if he finds anything.

Jay Pantone

I compted terms up to $n = 18$ and can't find any recurrence or guess a generating function. Let's think more about this.

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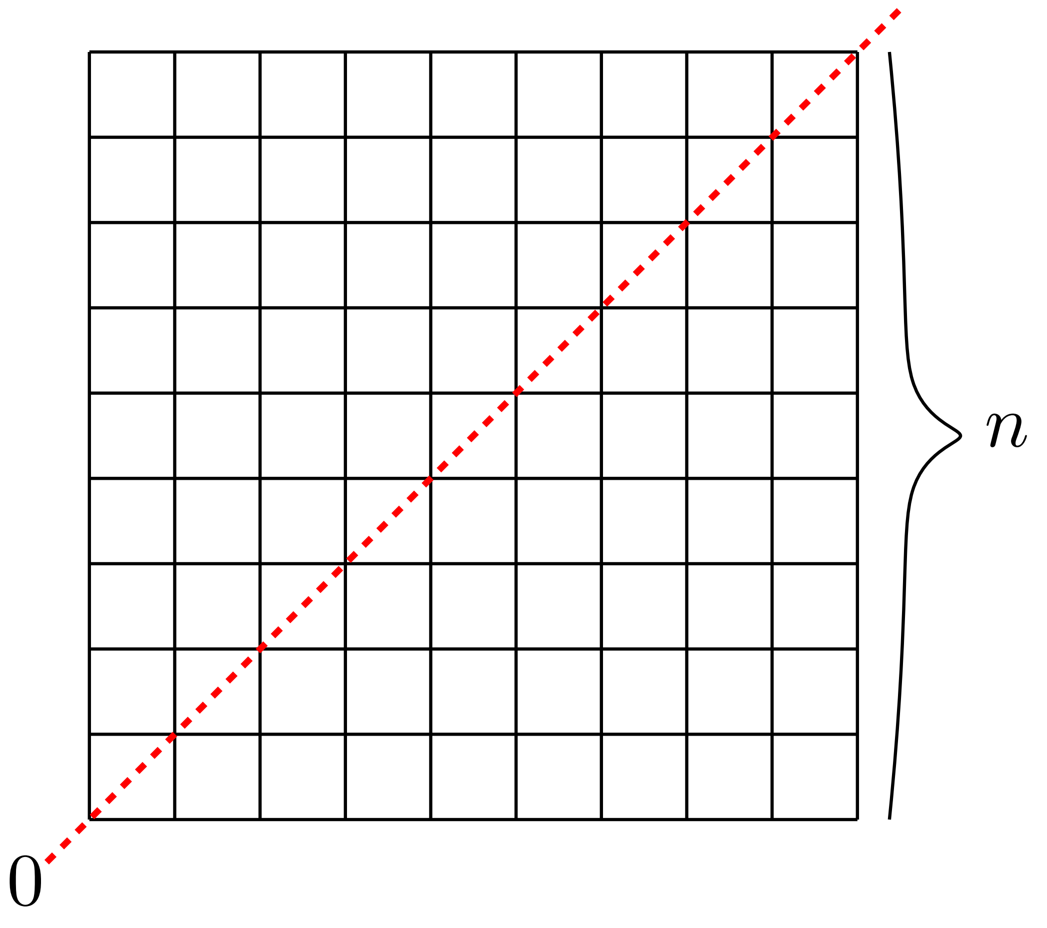
A new bivariate generating function

Definition

Given a set of permutations $\mathcal{C}$, the *diagonal generating function* for $\mathcal{C}$ is given by

$$C(x,q) = \sum a(n,k)x^n q^k$$

where $a(n,k)$ is the number of permutations in $\mathcal{C}$ of length n that have at least one point on diagonal k.



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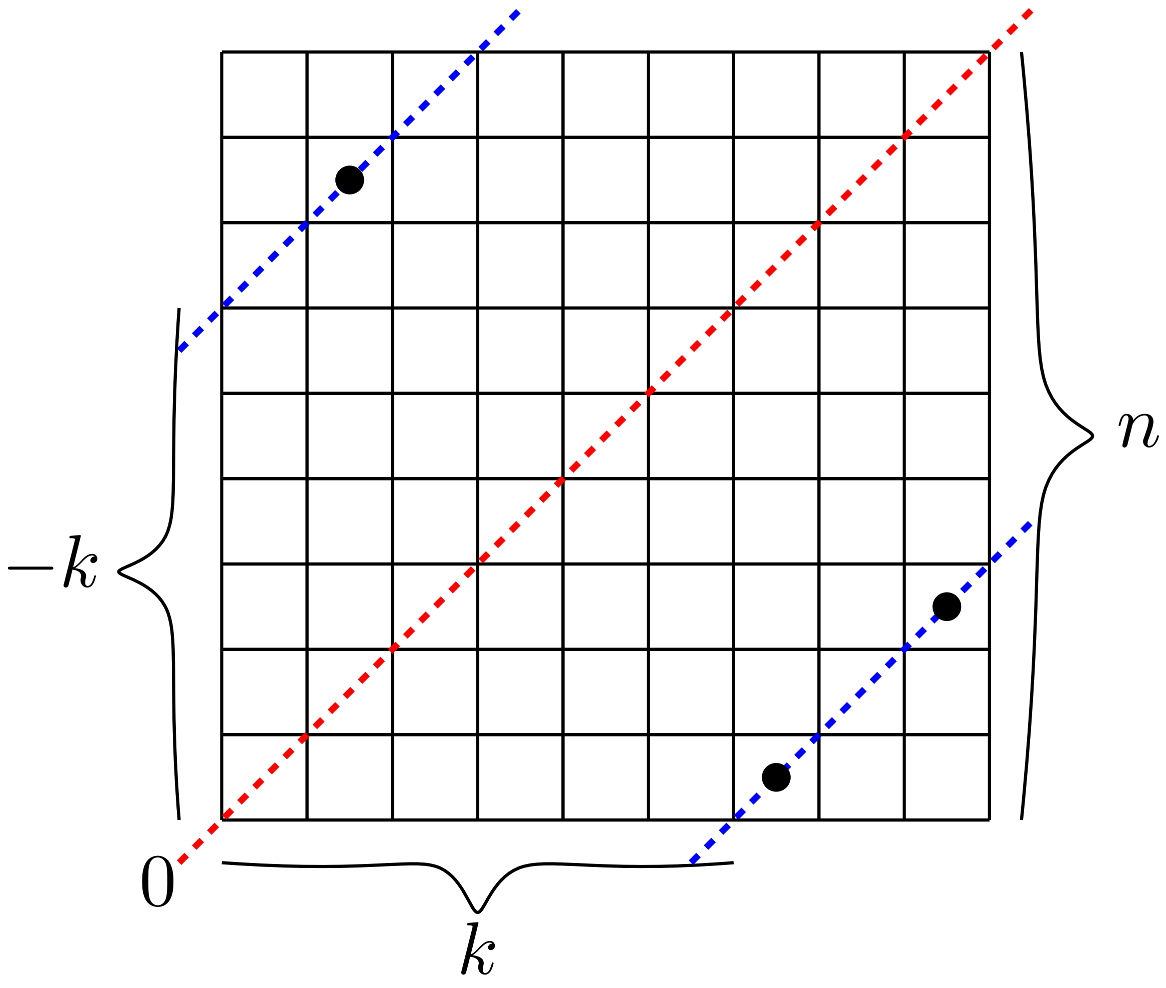
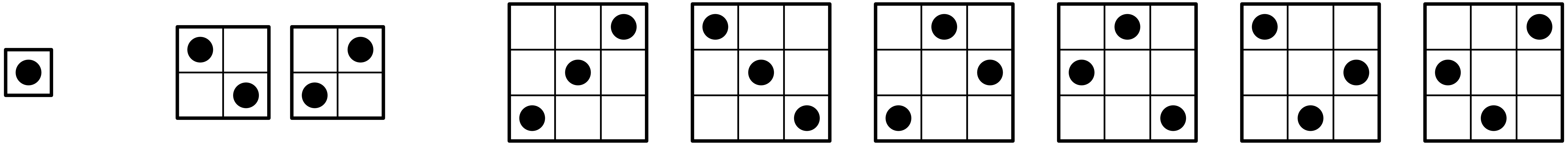


A new bivariate generating function

Example

Let $\mathcal{C} = \bigcup_{n \in \mathbb{N}} S_n$.



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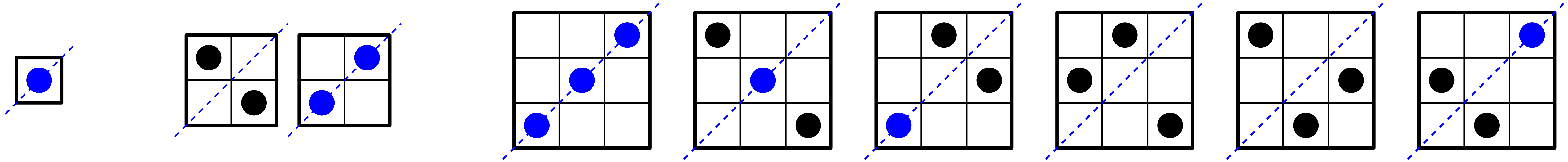

$x^1 q^0$ $\quad$ $x^2 q^0$ $\quad$ $x^3 q^0$ $\quad$ $x^3 q^0$ $\quad$ $x^3 q^0$ $\quad$ $x^3 q^0$

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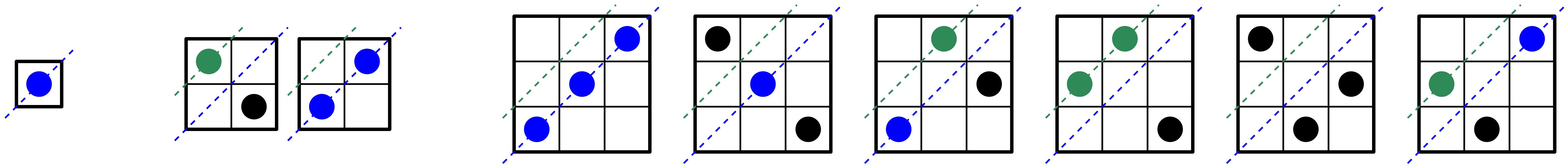

x^2q^{-1} x^3q^{-1} x^3q^{-1} x^3q^{-1}

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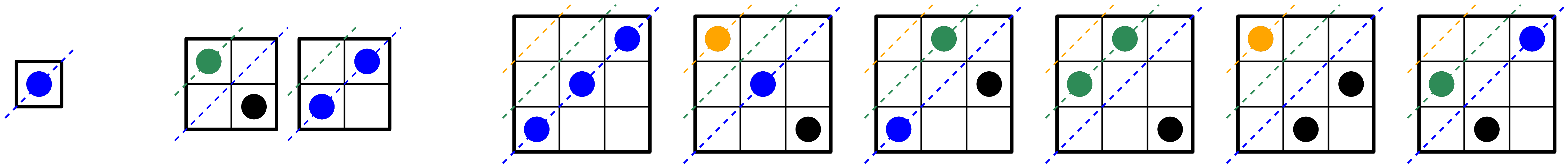

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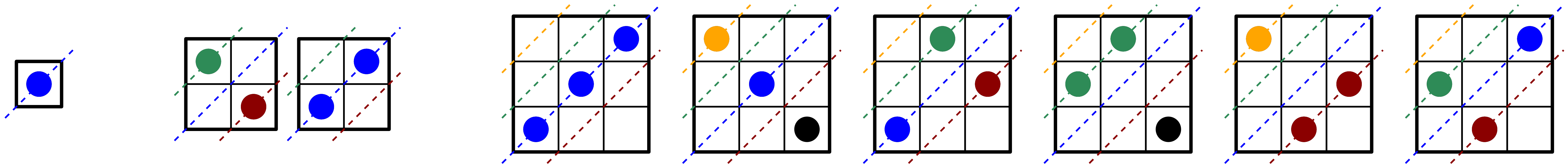

				x^3q^{-2}			x^3q^{-2}	
	x^2q^{-1}				x^3q^{-1}	x^3q^{-1}		x^3q^{-1}
x^1q^0		x^2q^0	x^3q^0	x^3q^0	x^3q^0			x^3q^0
	x^2q^1				x^3q^1		x^3q^1	x^3q^1

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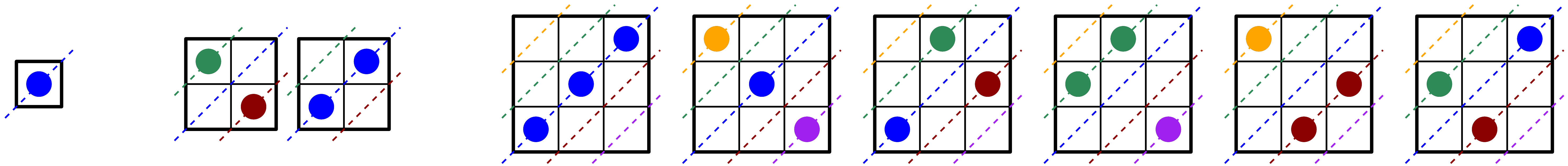

$x^3q^{-2} \quad x^3q^{-2}$

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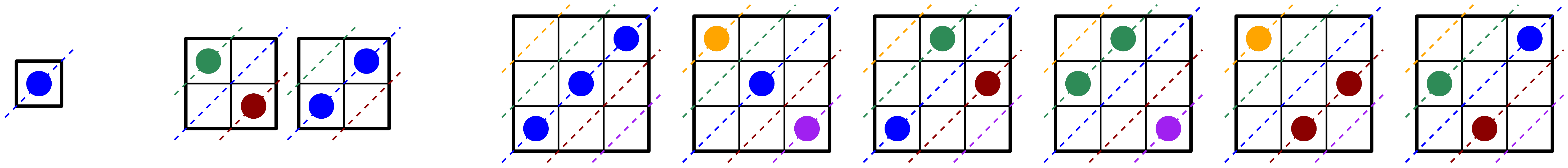

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x^2q^1 x^3q^1 x^3q^1 x^3q^1

x^3q^2 x^3q^2

$$C(x,q) = q^0x^1 + \left(2q^{-1} + q^0\right)x^2 + \left(2q^{-2} + 3q^{-1} + 4q^0 + 3q^1 + 2q^2\right)x^3 + \dots$$

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The diagonal generating function for $\mathcal{C}$ is NOT a refinement of the ordinary generating function for $\mathcal{C}$

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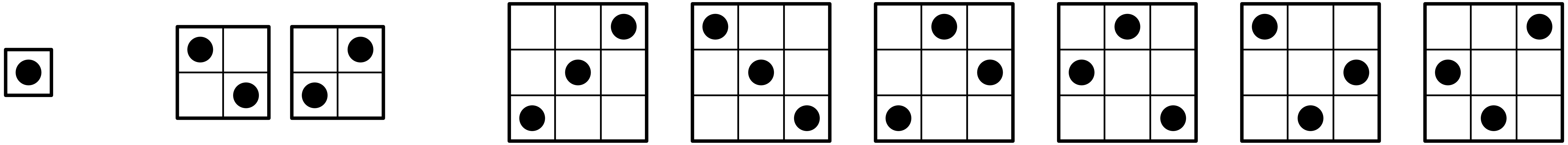

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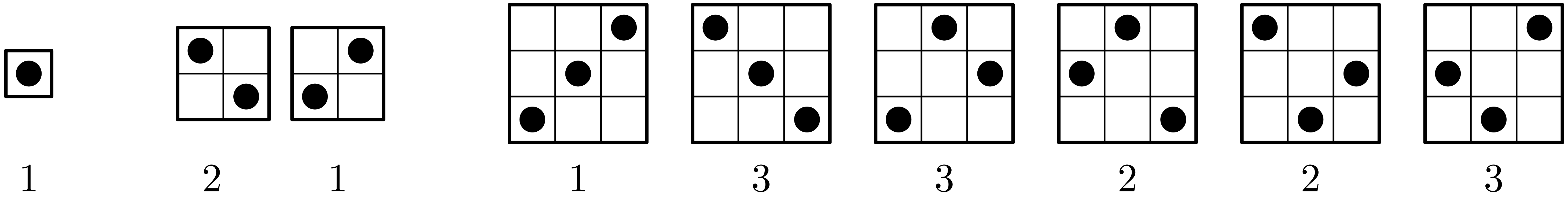

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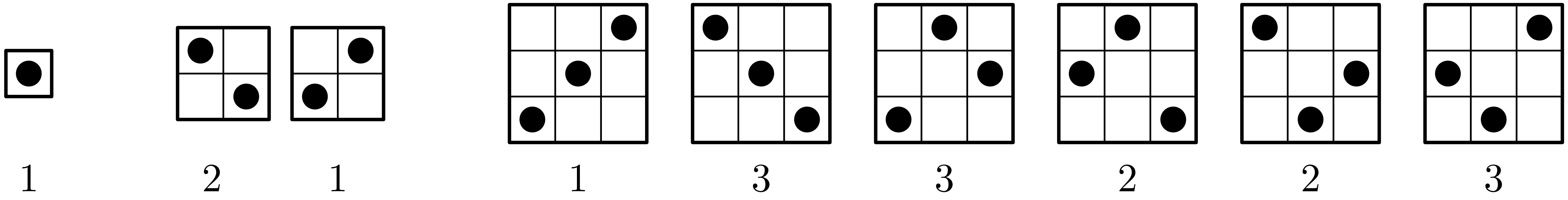

THE PLAN

$$C(x) - C(x,0) = \text{derangements in } \mathcal{C}$$

all permutations in $\mathcal{C}$

permutations in $\mathcal{C}$ with a fixed point

Detour: Separable Permutations

Theorem (Bose-Buss-Lubiw 1993)

The separable permutations (i.e. those that avoid 2413 and 3142) are precisely those that can be recursively decomposed as sums and skew-sums of smaller permutations.

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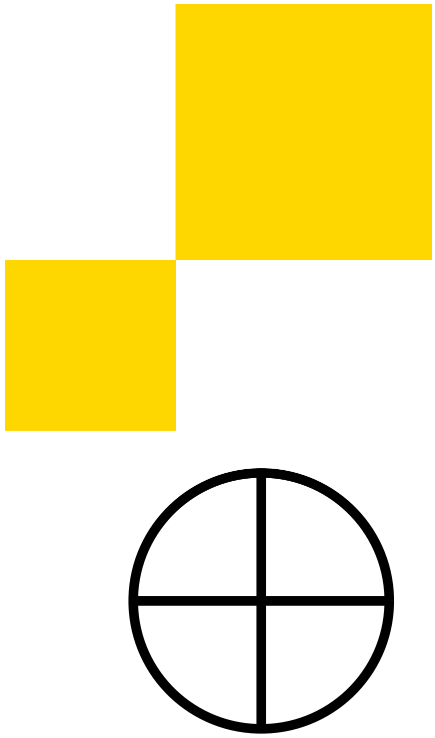
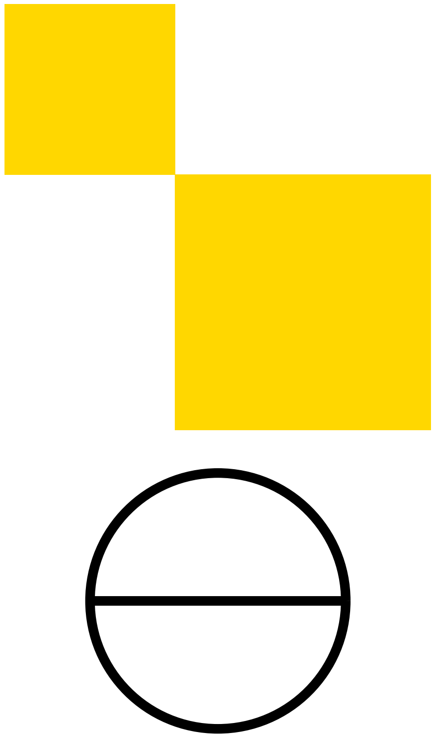


sum



skew-sum

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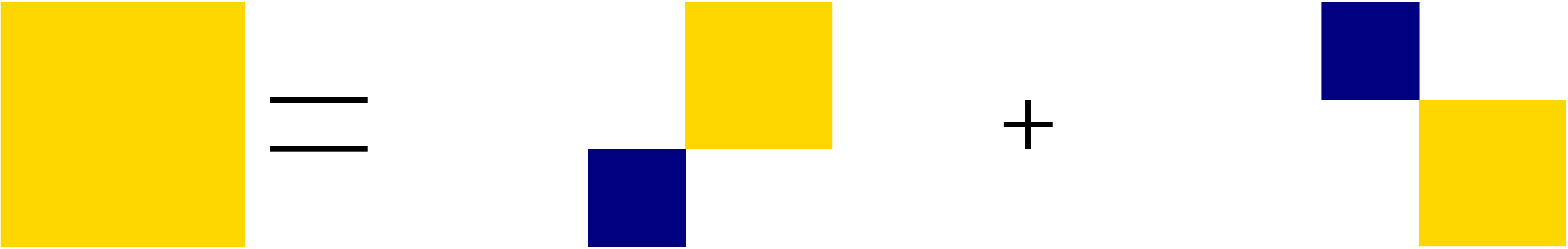

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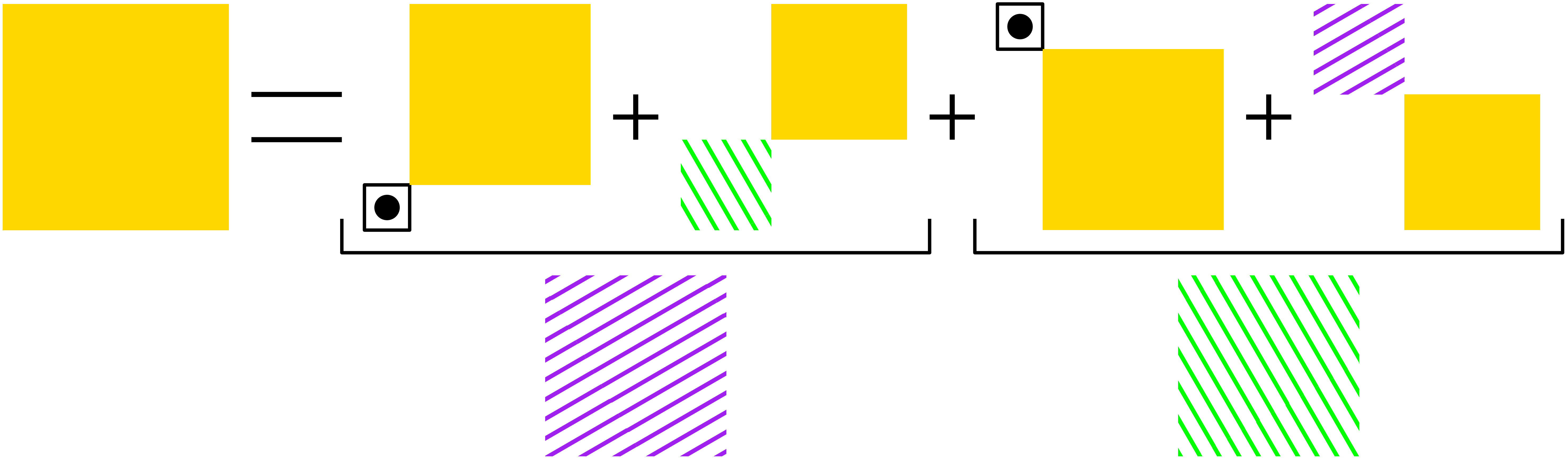

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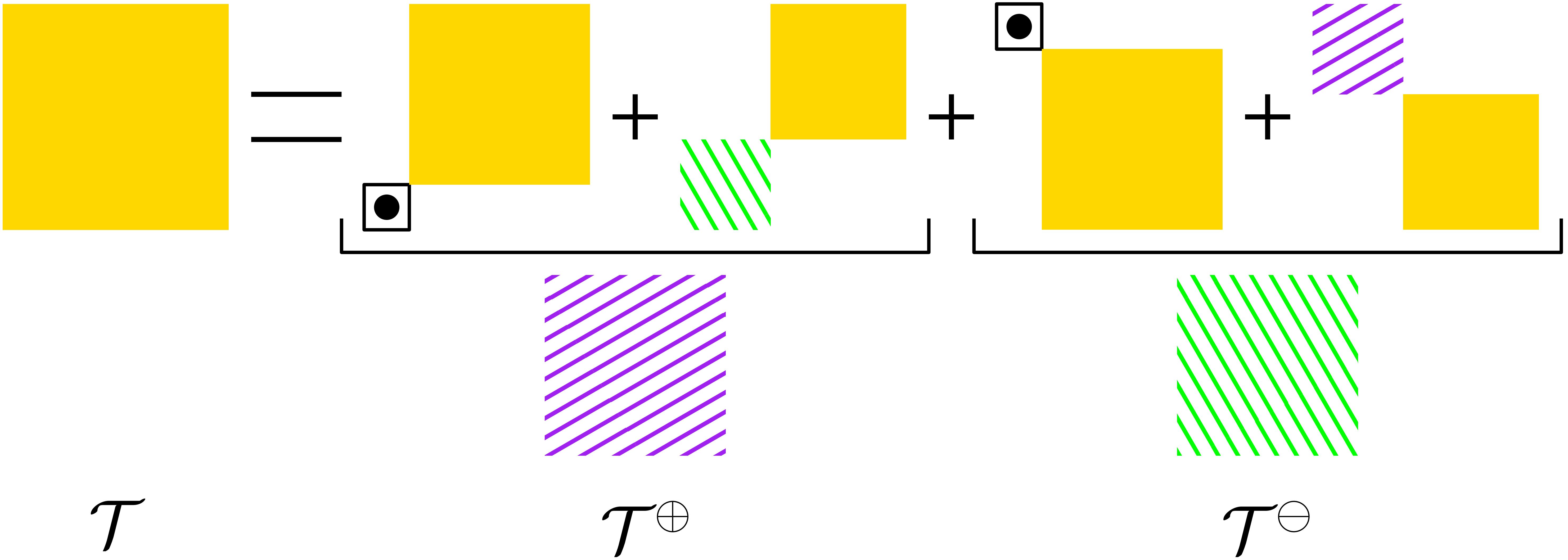
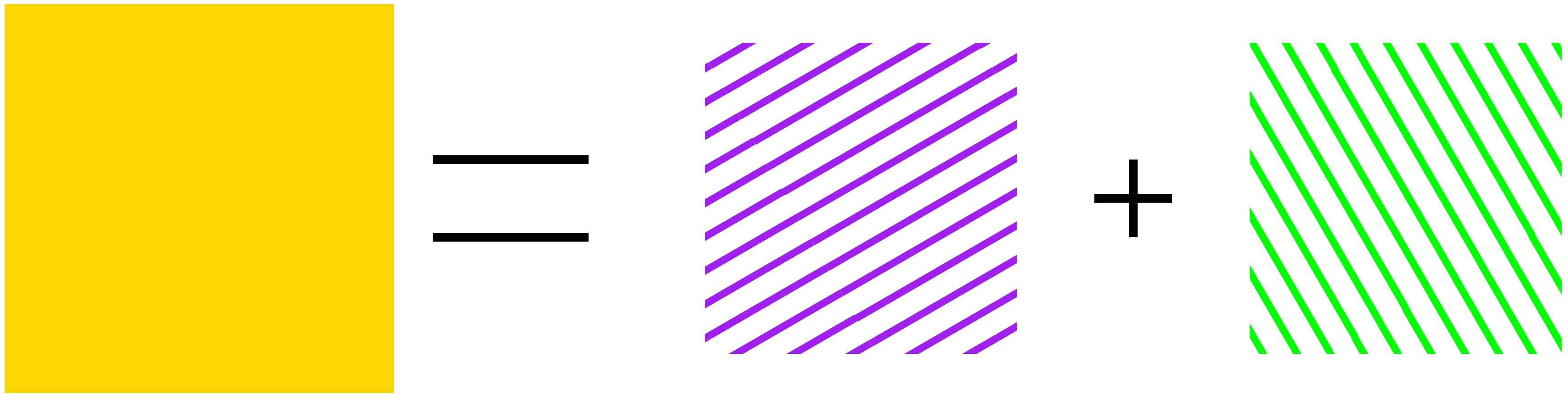


The main characters



$\mathcal{T}$ $\mathcal{T}^{\oplus}$ $\mathcal{T}^{\ominus}$

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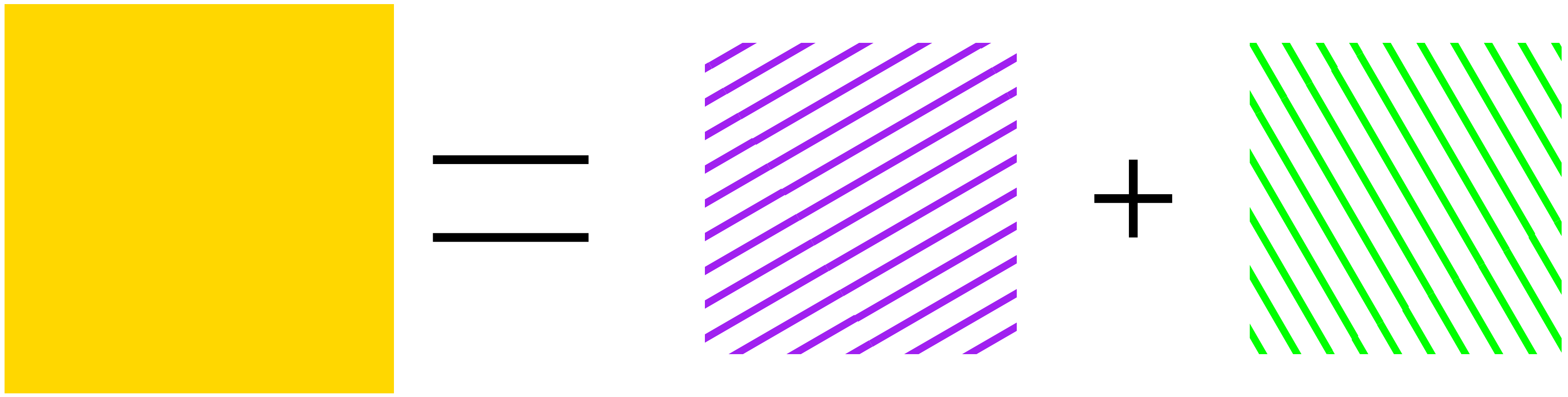

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ordinary generating function

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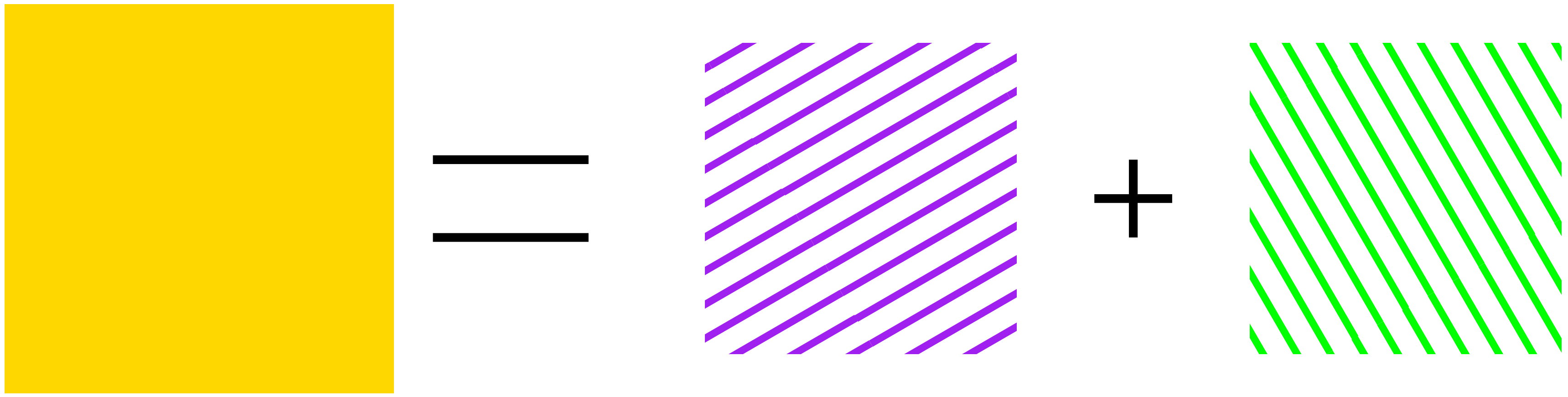

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$T(x)$ $\qquad$ $T^{\oplus}(x)$ $\qquad$ $T^{\ominus}(x)$

$T(x,q)$ $\qquad$ $T^{\oplus}(x,q)$ $\qquad$ $T^{\ominus}(x,q)$

diagonal generating function

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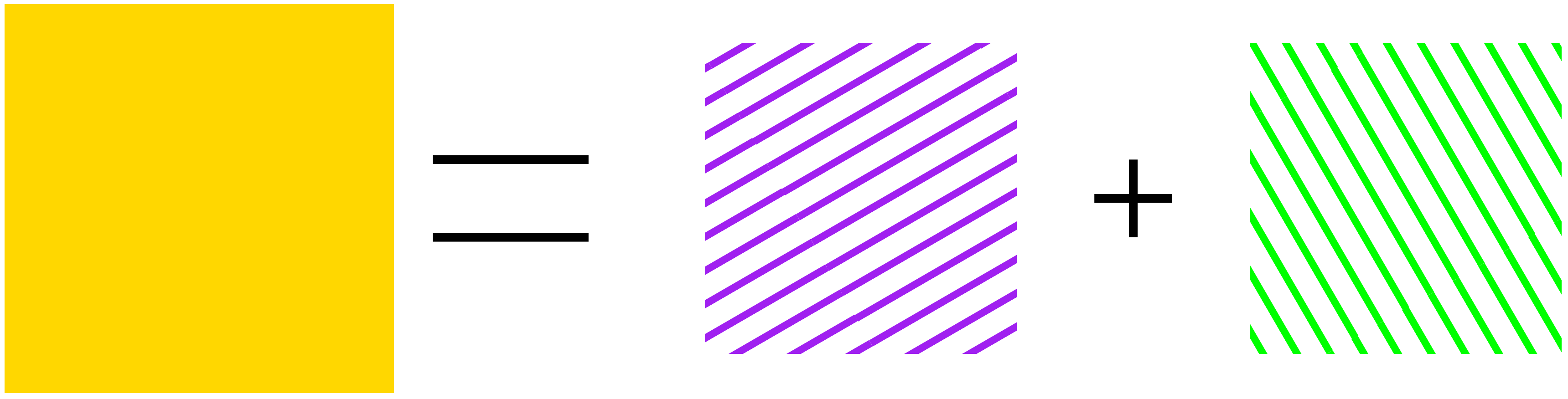


$$\mathcal{T} = \mathcal{T}^{\oplus} + \mathcal{T}^{\ominus}$$

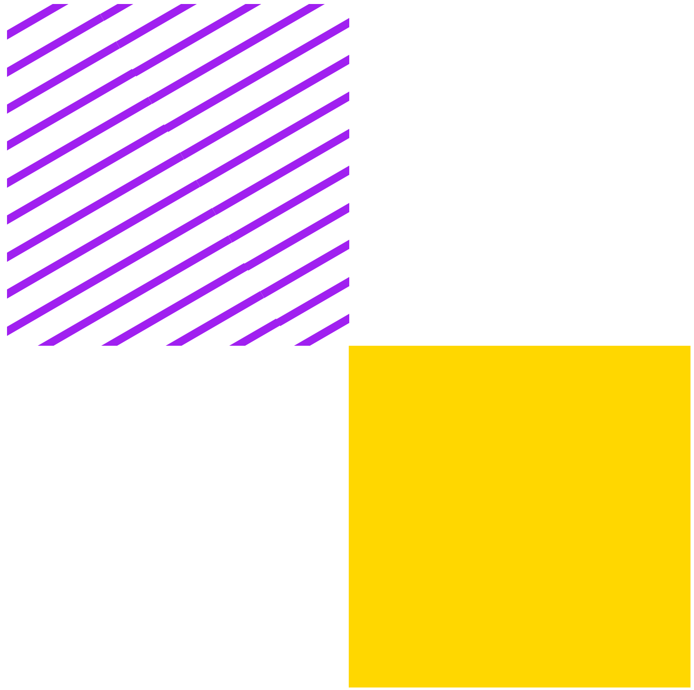
ordinary generating function

$$T(x) = T^{\oplus}(x) + T^{\ominus}(x)$$

$$T(x,q) = T^{\oplus}(x,q) + T^{\ominus}(x,q)$$

diagonal generating function

Skew-sum decomposable permutations



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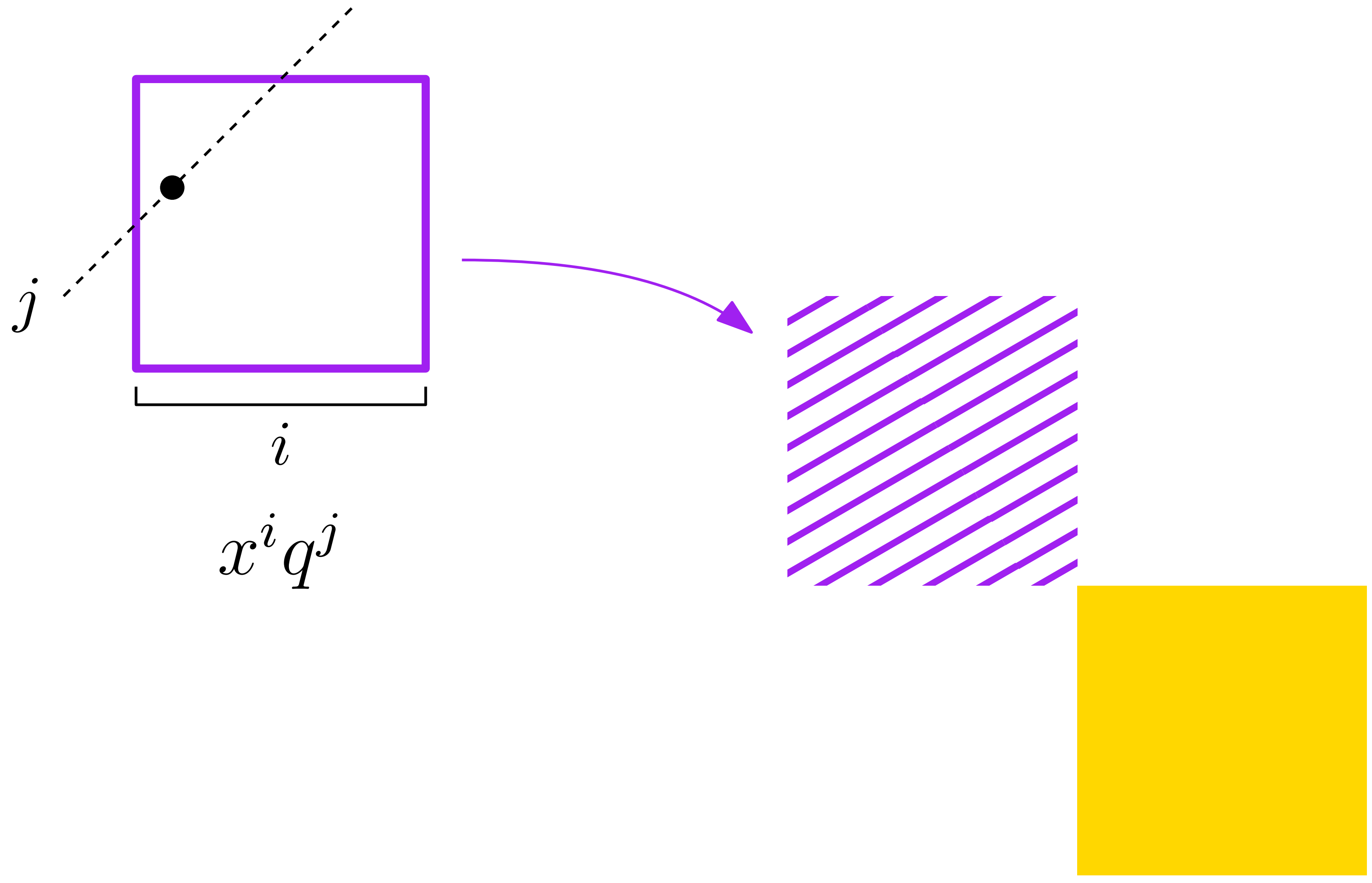

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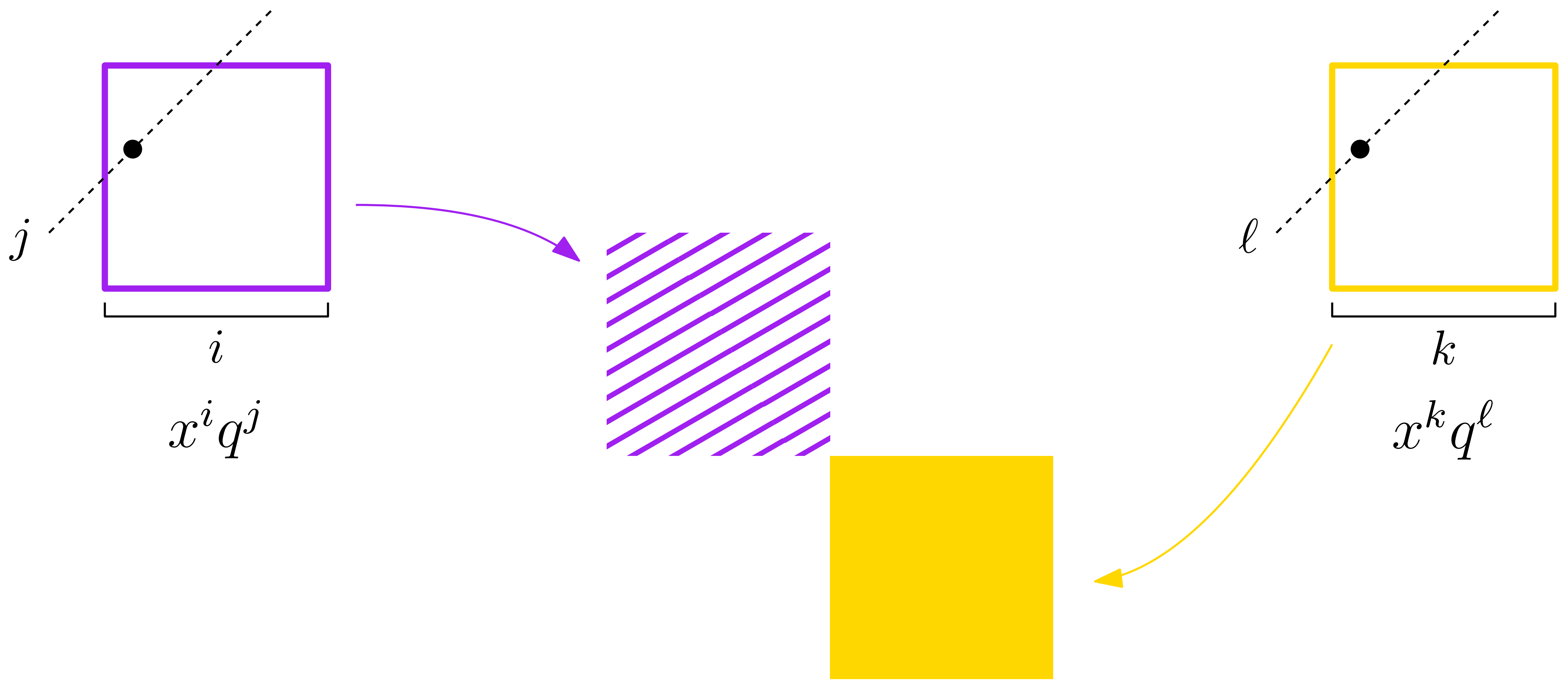

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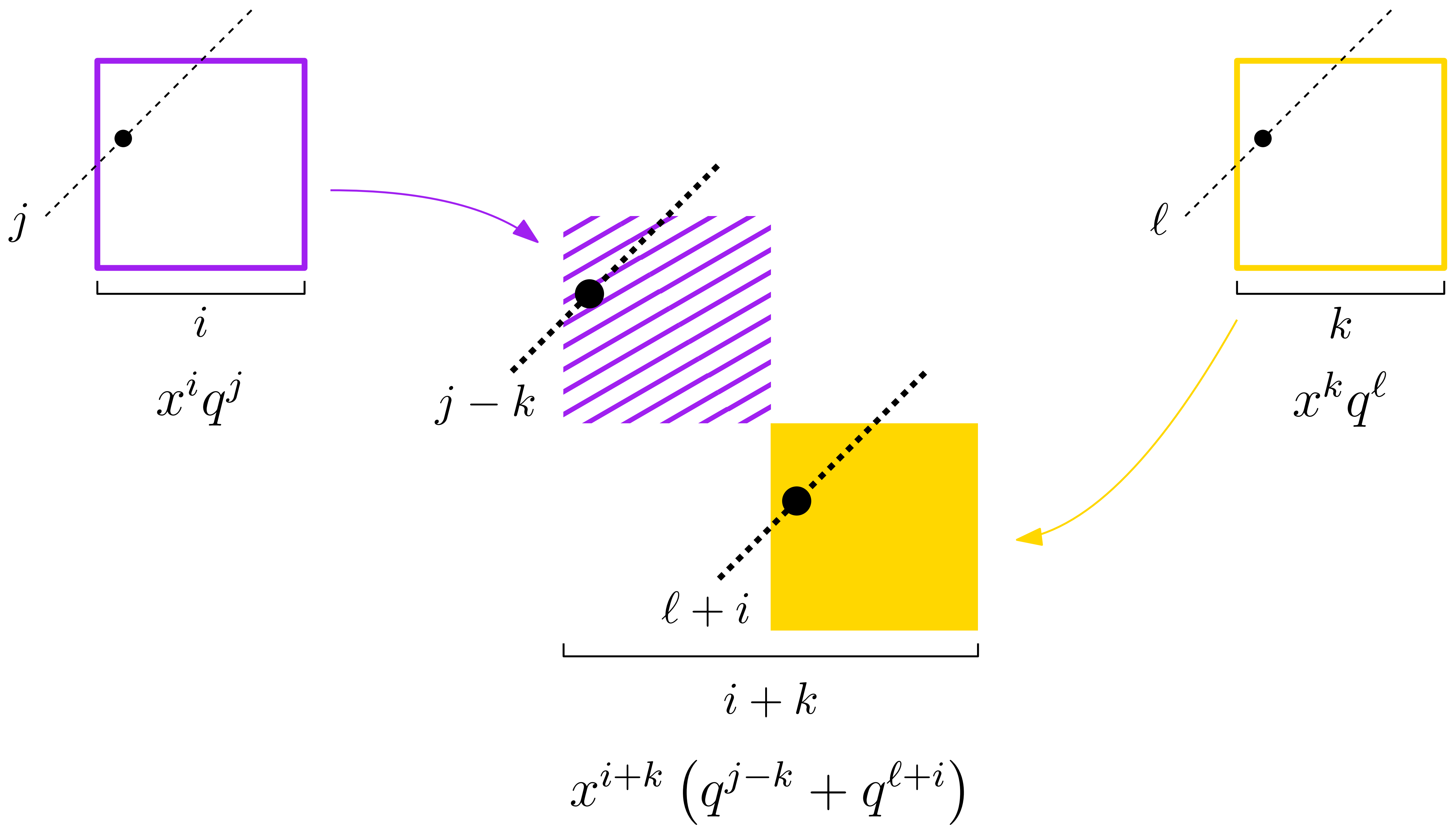

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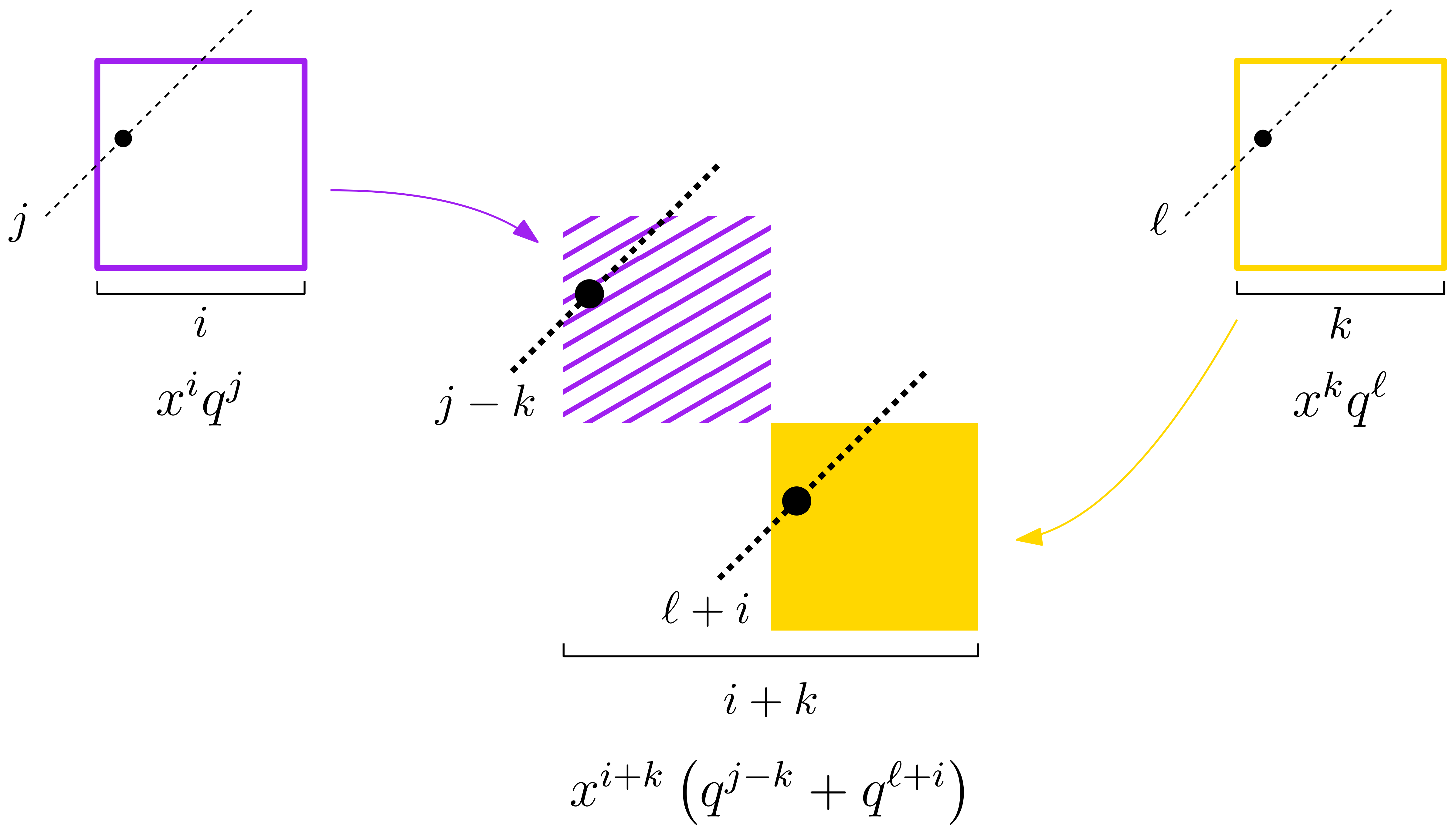
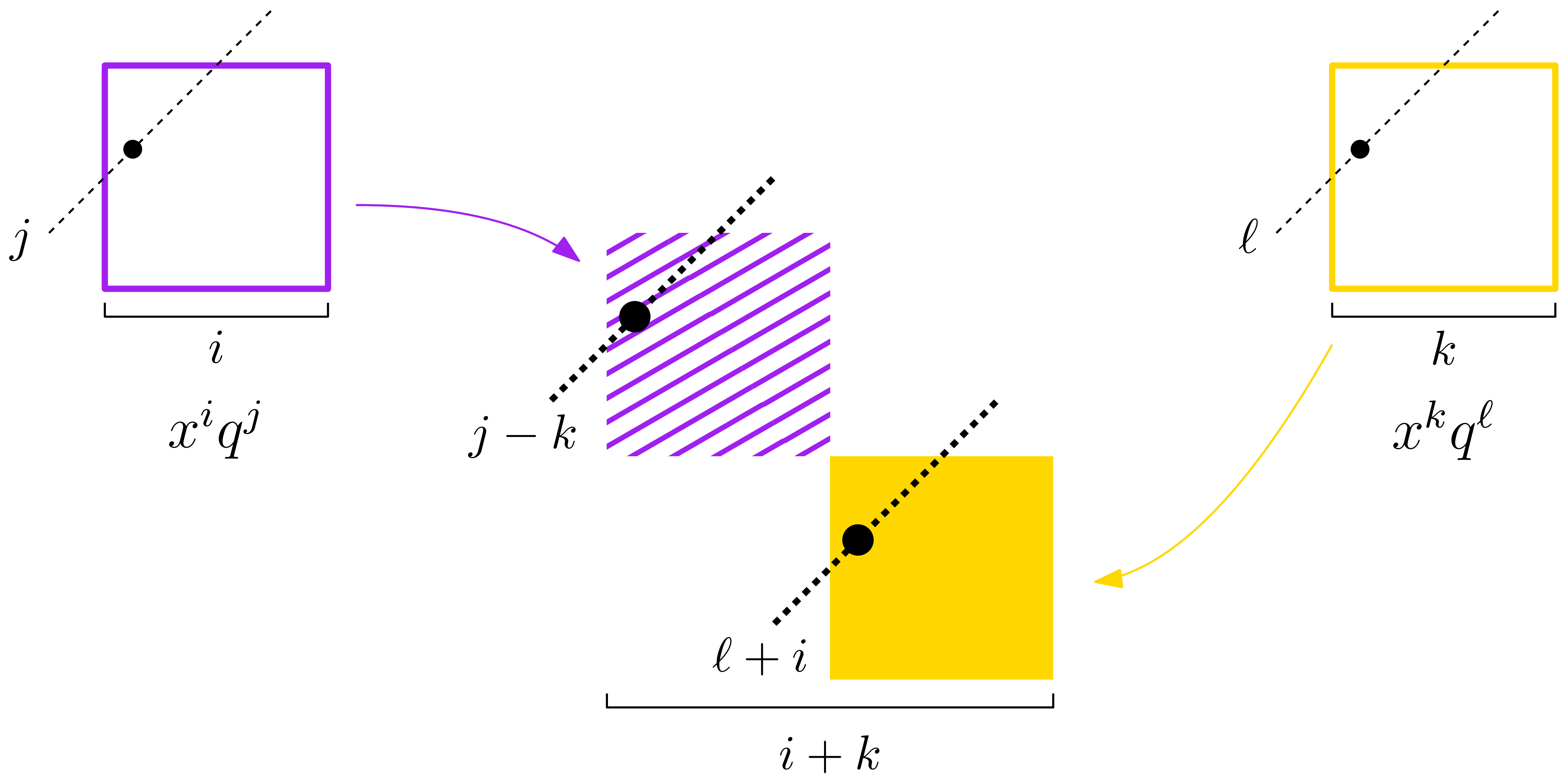


$$T^{\ominus}(x,q) = T^{\oplus}(x,q)\, T\left(\frac{x}{q}\right) + T^{\oplus}(xq)\, T(x,q)$$

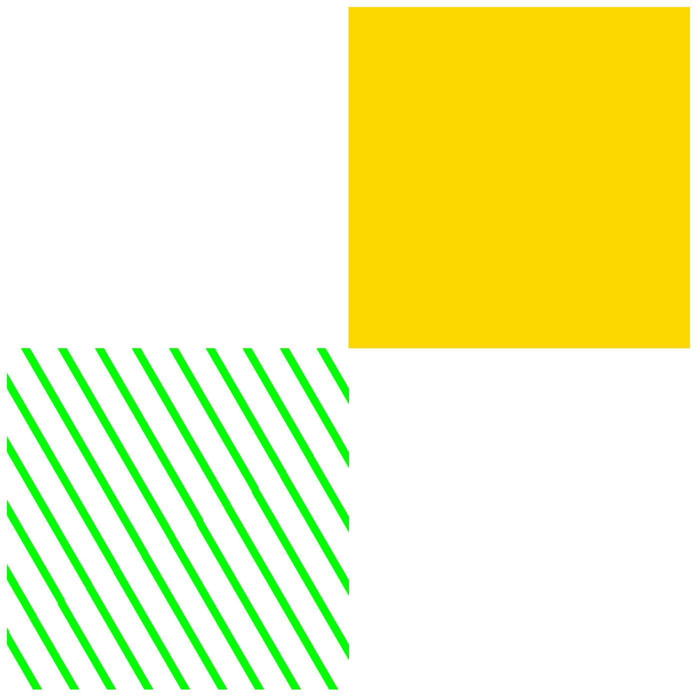
Skew-sum decomposable permutations



$$x^{i+k}\left(q^{j-k}+q^{\ell+i}\right)$$

$$T^{\ominus}(x,q) \;\dot{=}\; \left(xq^0 + T^{\oplus}(x,q)\right) T\left(\frac{x}{q}\right) + \left(xq + T^{\oplus}(xq)\right) T(x,q)$$

Sum decomposable permutations



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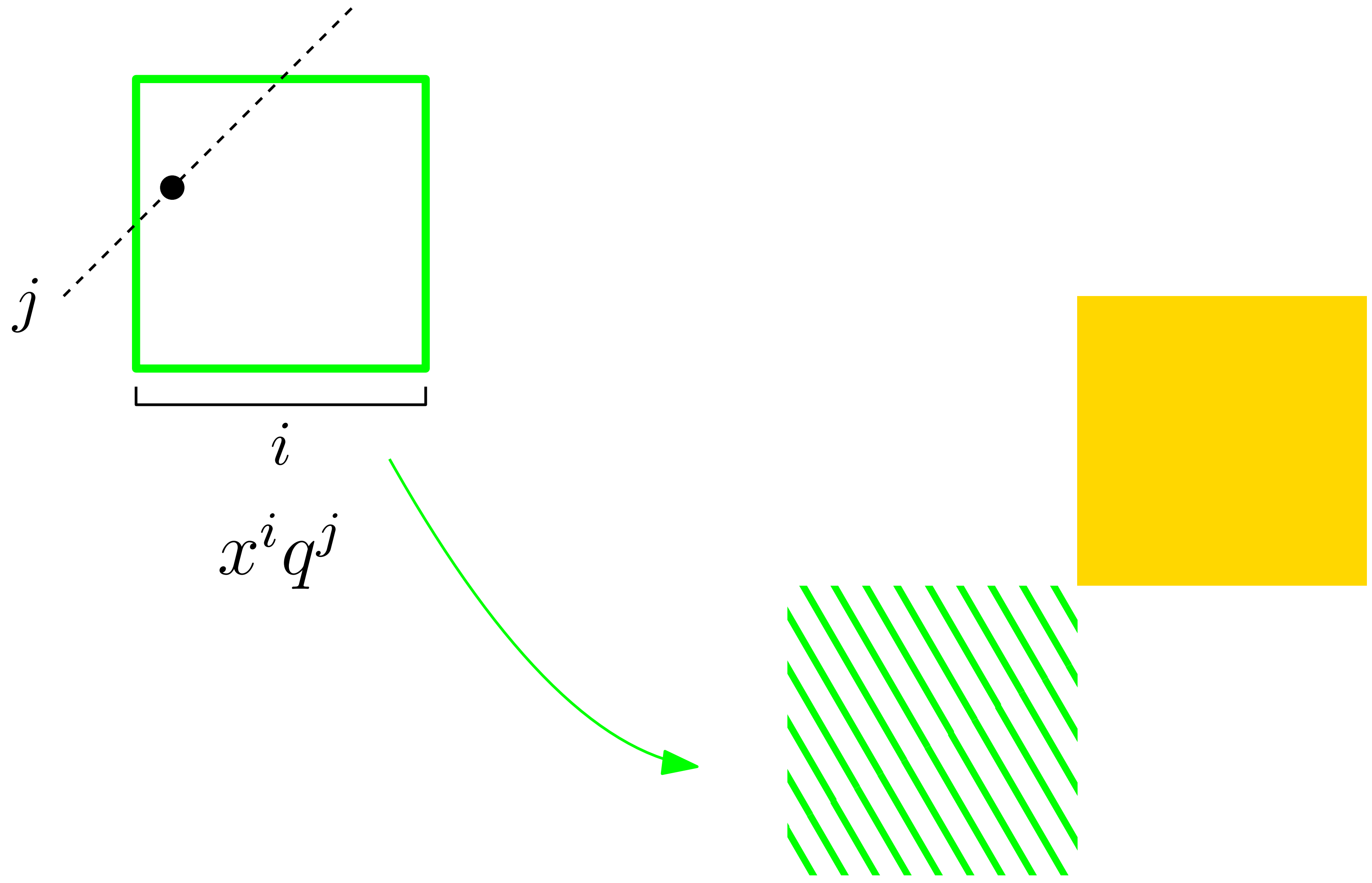

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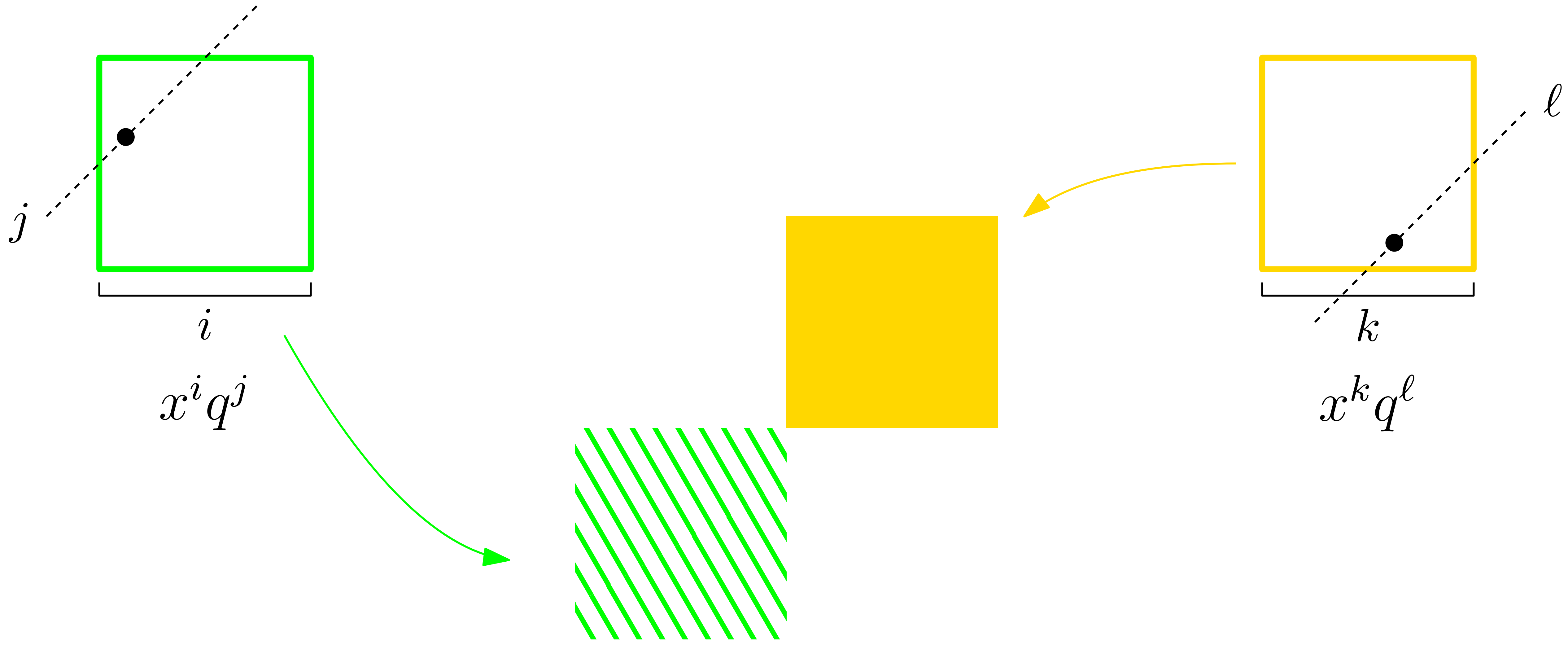

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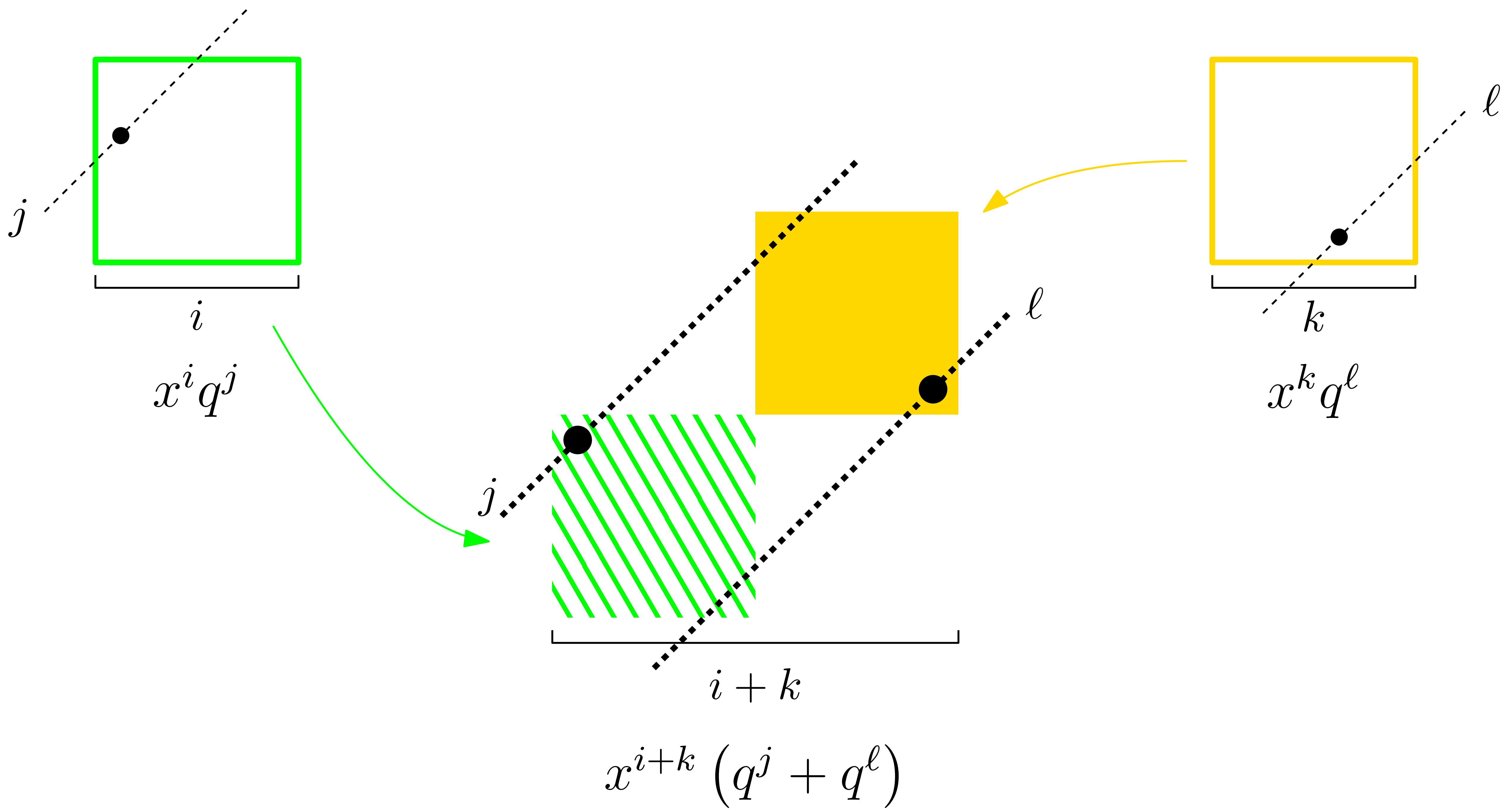

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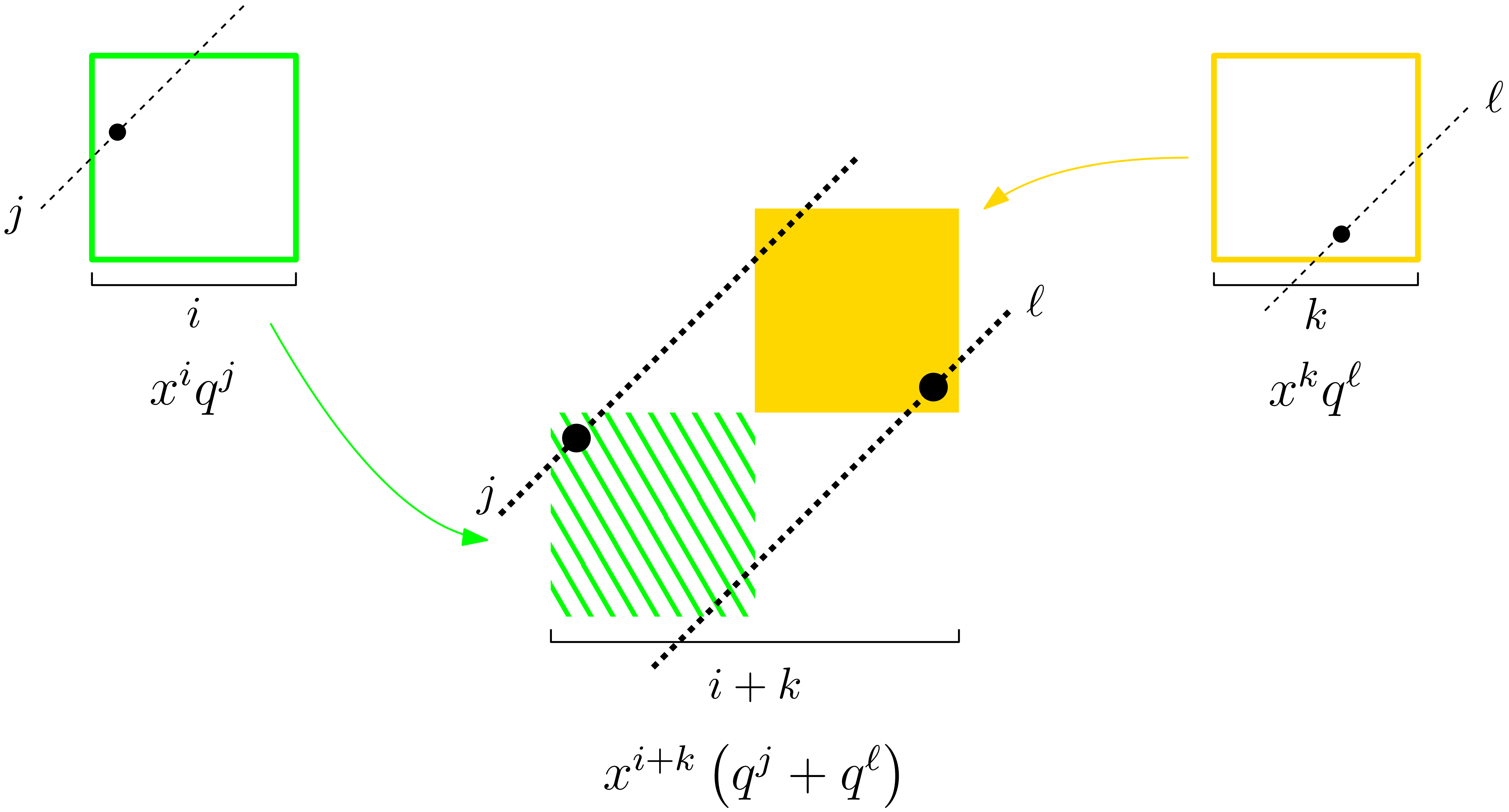

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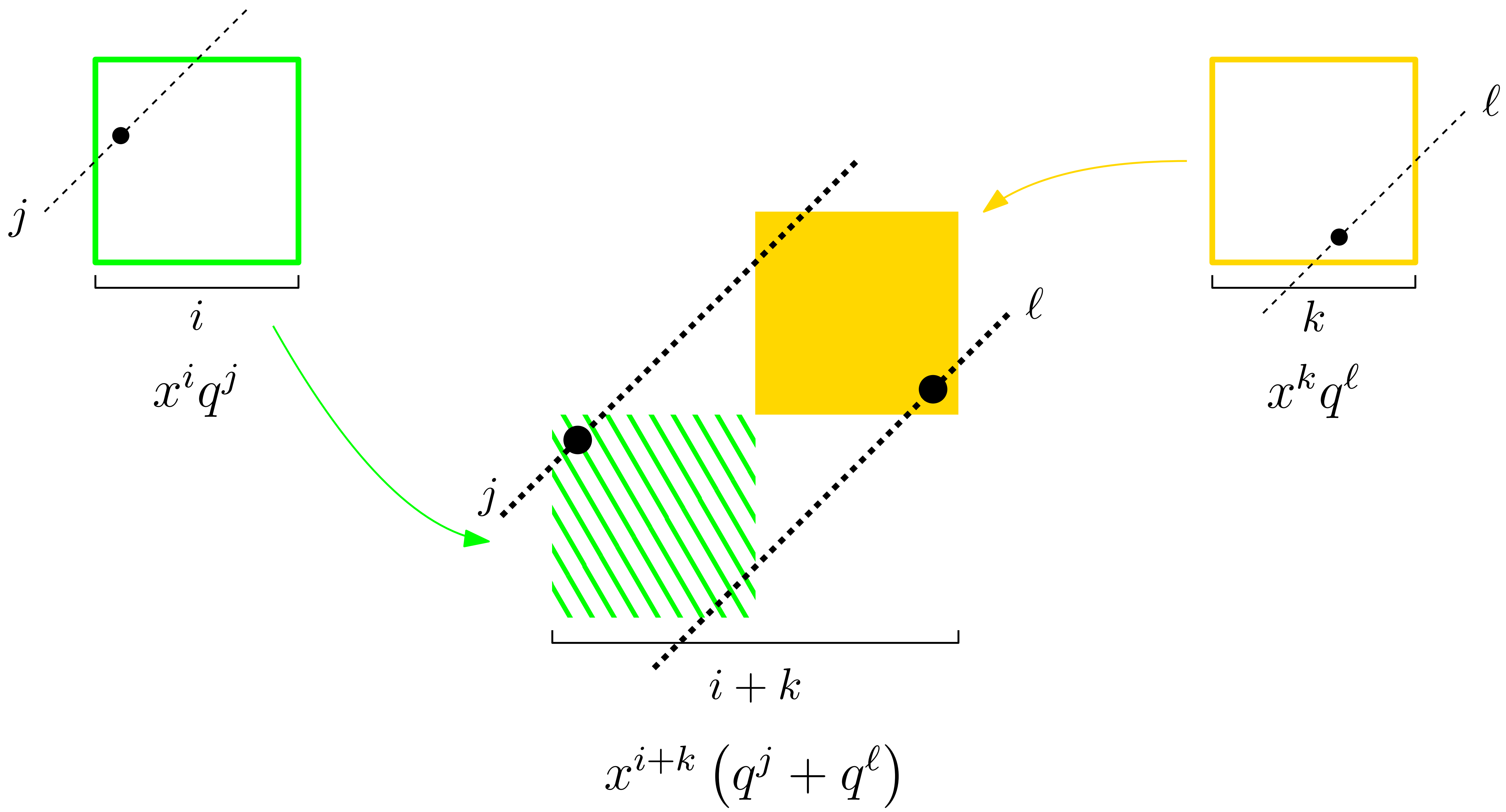
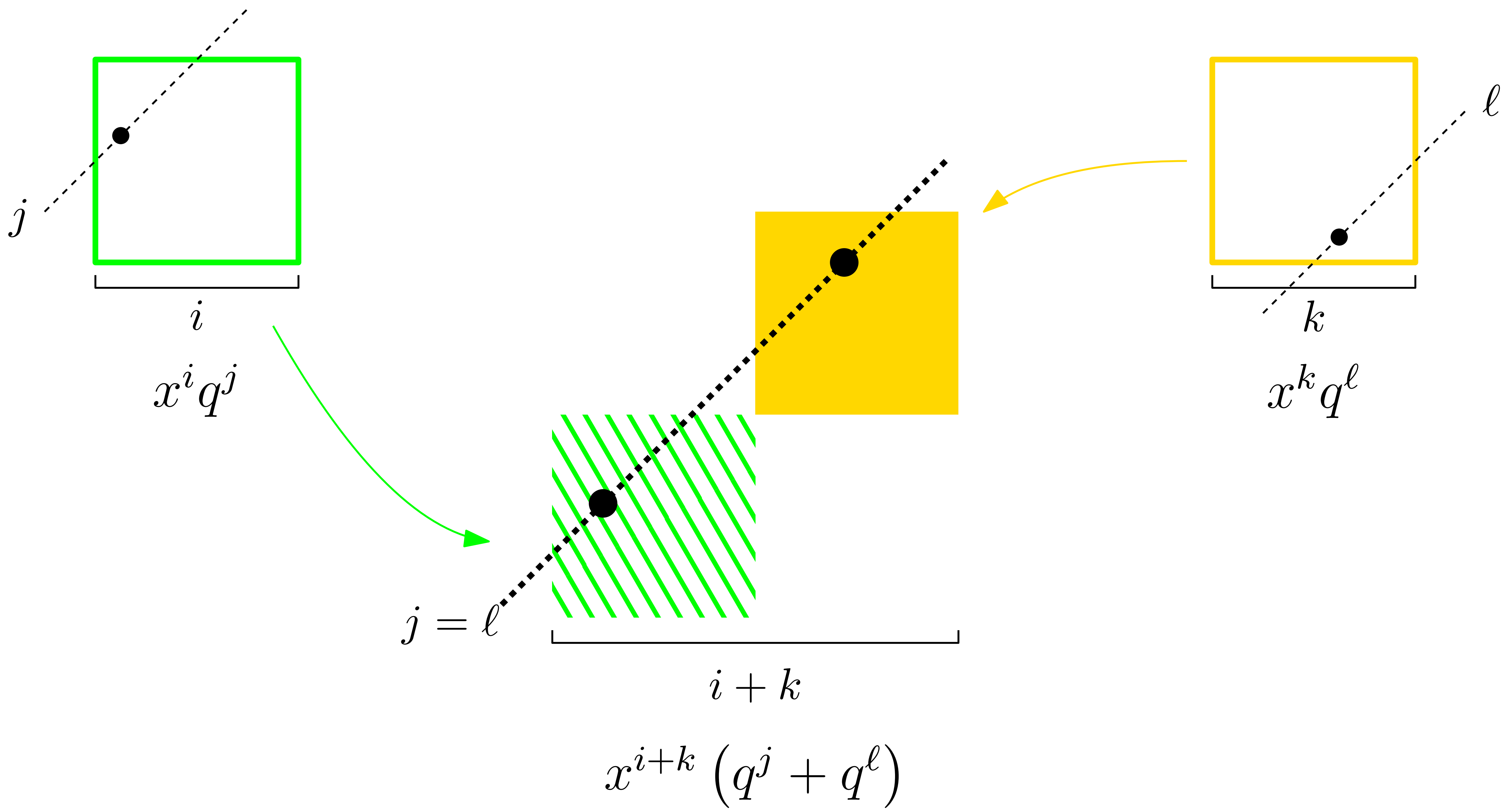


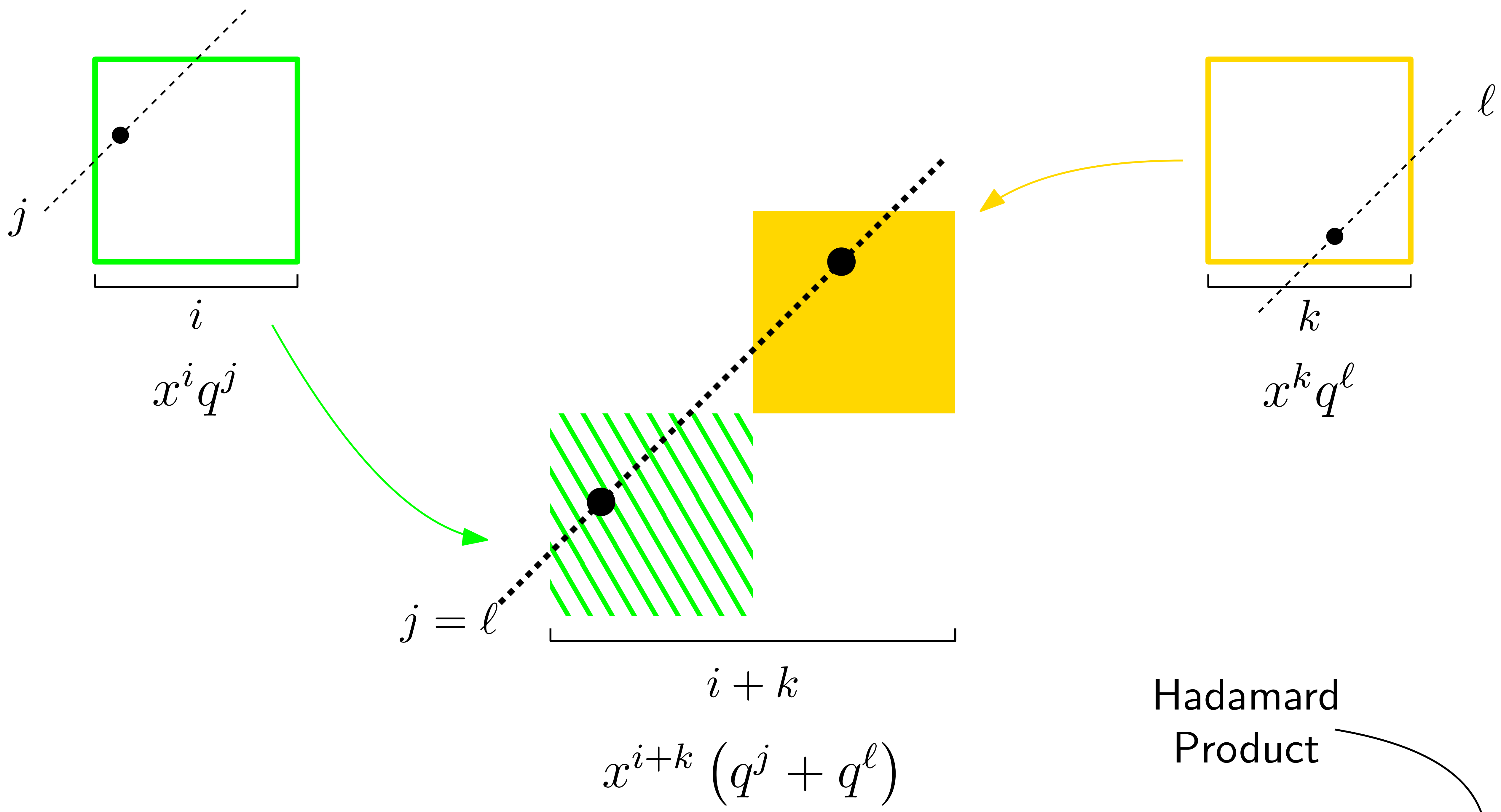
$$T^{\oplus}(x,q) = \left(xq^0 + T^{\ominus}(x,q)\right) T(x) + \left(x + T^{\ominus}(x)\right) T(x,q)$$

Sum decomposable permutations



$$T^{\oplus}(x,q) \doteq \left(xq^0 + T^{\ominus}(x,q)\right) T(x) + \left(x + T^{\ominus}(x)\right) T(x,q)$$

Sum decomposable permutations



$$T^{\oplus}(x,q) \doteq \left(xq^0 + T^{\ominus}(x,q)\right) T(x) + \left(x + T^{\ominus}(x)\right) T(x,q) - \left(xq^0 + T^{\ominus}(x,q)\right) \underset{q}{\odot} T(x,q)$$

The grand finale!

$$T(x,q) \quad = \; xq^0 + T^{\oplus}(x,q) + T^{\ominus}(x,q)$$

The grand finale!

$$T(x,q) \doteq xq^0 + T^{\oplus}(x,q) + T^{\ominus}(x,q)$$

$$T^{\ominus}(x,q) \doteq \left(xq^0 + T^{\oplus}(x,q)\right) T\left(\frac{x}{q}\right) + \left(xq + T^{\oplus}(xq)\right) T(x,q)$$

The grand finale!

$$T(x,q) \doteq xq^0 + T^{\oplus}(x,q) + T^{\ominus}(x,q)$$

$$T^{\ominus}(x,q) \doteq \left(xq^0 + T^{\oplus}(x,q)\right) T\left(\frac{x}{q}\right) + \left(xq + T^{\oplus}(xq)\right) T(x,q)$$

$$T^{\oplus}(x,q) \doteq \left(xq^0 + T^{\ominus}(x,q)\right) T(x) + \left(x + T^{\ominus}(x)\right) T(x,q) - \left(xq^0 + T^{\ominus}(x,q)\right) \underset{q}{\bigodot} T(x,q)$$

The grand finale!

$$T(x,q) = xq^0 + T^{\oplus}(x,q) + T^{\ominus}(x,q)$$

$$T^{\ominus}(x,q) = \left(xq^0 + T^{\oplus}(x,q)\right) T\left(\frac{x}{q}\right) + \left(xq + T^{\oplus}(xq)\right) T(x,q)$$

$$T^{\oplus}(x,q) = \left(xq^0 + T^{\ominus}(x,q)\right) T(x) + \left(x + T^{\ominus}(x)\right) T(x,q) - \left(xq^0 + T^{\ominus}(x,q)\right) \underset{q}{\odot} T(x,q)$$

$\underset{\text{Jay}}{\rightsquigarrow}$ 1000 terms of the sequence enumerating separable derangements

The grand finale!

$$
\begin{aligned}
T(x,q) &= xq^0 + T^{\oplus}(x,q) + T^{\ominus}(x,q) \\
T^{\ominus}(x,q) &= \left(xq^0 + T^{\oplus}(x,q)\right) T\left(\frac{x}{q}\right) + \left(xq + T^{\oplus}(xq)\right) T(x,q) \\
T^{\oplus}(x,q) &= \left(xq^0 + T^{\ominus}(x,q)\right) T(x) + \left(x + T^{\ominus}(x)\right) T(x,q) - \left(xq^0 + T^{\ominus}(x,q)\right) \bigodot_q T(x,q)
\end{aligned}
$$

$\xrightarrow[\text{Jay}]{}$ 1000 terms of the sequence enumerating separable derangements

❌ Recurrence

❌ Closed Form

The grand finale!

$$T(x,q) = xq^0 + T^{\oplus}(x,q) + T^{\ominus}(x,q)$$

$$T^{\ominus}(x,q) = \left(xq^0 + T^{\oplus}(x,q)\right) T\left(\frac{x}{q}\right) + \left(xq + T^{\oplus}(xq)\right) T(x,q)$$

$$T^{\oplus}(x,q) = \left(xq^0 + T^{\ominus}(x,q)\right) T(x) + \left(x + T^{\ominus}(x)\right) T(x,q) - \left(xq^0 + T^{\ominus}(x,q)\right) \underset{q}{\odot} T(x,q)$$

$\underset{\text{Jay}}{\rightsquigarrow}$ 1000 terms of the sequence enumerating separable derangements

✗ Recurrence

✗ Closed Form

Conjecture

This sequence is not D-finite.

References

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